On the L^p Spectrum

Nelia Charalambous

Joint work with Zhiqin Lu and Julie Rowlett

Department of Mathematics and Statistics
University of Cyprus

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Introduction

On a smooth Riemannian manifold M^n the Laplacian on k-forms is given by

$$\Delta = d\,\delta + \delta d = \nabla^* \nabla + \mathcal{W}_k$$

$\mathcal{W}_k$ is the curvature (Weitzenböck) tensor on k-forms. It coincides with Ric when $k = 1$ and is bounded below when the curvature tensor is bounded below.

Δ is a self-adjoint, nonnegative definite operator on $L^2(\Lambda^k)$.
It can also be extended to a closed operator on $L^p(\Lambda^k)$.

The *spectrum* of Δ consists of all points $\lambda \in \mathbb{C}$ for which

$$\Delta - \lambda I$$

fails to be invertible on its domain.

The definition for the spectrum is the same for any self-adjoint operator H on a Hilbert space $\mathcal{H}$, and also for a closed operator on a Banach space.

The L^p spectrum

• The semigroup operator associated to the Laplacian $e^{-t\Delta}$ can be extended to a bounded operator on the Banach space $L^p(\Lambda^k)$ for $1 \leq p \leq \infty$ whenever $\mathcal{W}_k$ is bounded below ($L^\infty = (L^1)^*$).

• The Laplacian on $L^p(\Lambda^k)$ is defined as the generator of this semigroup.

• Its spectrum on $L^p(\Lambda^k)$ is defined as above. We denote it by $\sigma(p,k)$, and note that it can depend on both p and k. By duality, we have

$$\sigma(p,k) = \sigma(p^*,k) \;\text{ whenever }\; \frac{1}{p} + \frac{1}{p^*} = 1.$$

Poincaré duality gives: $\sigma(p,k) = \sigma(p,n-k)$.

• The complement of the spectrum is known as the resolvent set $\rho(p,k)$.
It is the set of points $\mu \in \mathbb{C}$ where $(\Delta - \mu)^{-1}$ is bounded.

• For $p \neq 2$, since the operator need no longer be self-adjoint, its spectrum can contain nonreal eigenvalues.

The L^p spectrum

• The spectrum of geometric operators like the Laplacian reflects the geometric structure and analytic properties of the space and the bundle.

• When the manifold is compact, the L^p spectrum is always a discrete set and is the same for all p. Therefore, studying the L^p spectrum for $p \neq 2$ is only interesting in the noncompact case.

• It is possible to prove the L^p independence of the spectrum for the Laplacian (and Schrödinger operators) under certain conditions on the curvature and geometry of the manifold.

- Hempel and Voigt 1986-87: Schrödinger operators over $\mathbb{R}^n$ depending on the growth of the negative part of the potential.

- Sturm 1992: Uniformly elliptic operators in divergence form (including the Laplacian on functions) over manifolds with Ricci curvature bounded below and uniformly subexponential volume growth.

- C -N. Grosse 2023: Dirac operators over Clifford bundles if, in addition to Sturm's geometric conditions, the Clifford contraction (Weitzenböck tensor for k-forms) is bounded below.

The L^p spectrum

• On the other hand, $\sigma(p,k)$ may contain a region in the complex plane when the curvature is negative at an end of the manifold.

• It was proved by Davies, Simon and Taylor 1988 that over the hyperbolic space $\mathbb{H}^n$, the spectrum of the Laplacian on functions depends on p. $\sigma(p,0)$ is the parabolic region in the complex plane

$$Q_{p,0} = \left\{ \frac{(n-1)^2}{4} + z^2 \,\middle|\, z \in \mathbb{C} \text{ with } |\operatorname{Im} z| \leq (n-1)\left|\frac{1}{p} - \frac{1}{2}\right| \right\}.$$

The parabolic region reduces to the interval $[(n-1)^2/4, \infty) \subset \mathbb{R}$ when $p = 2$, and to the region $\{z \mid \operatorname{Re} z \geq (\operatorname{Im} z/(n-1))^2\}$ when $p = 1$.

• They showed that for Kleinian Groups, $M = \mathbb{H}^n/\Gamma$, with Γ a geometrically finite group of isometries of $\mathbb{H}^n$ such that M is either cusp-free or of finite volume, the L^p spectrum is the same parabolic region together with a finite number of isolated eigenvalues on the real line, which are L^2 eigenvalues.

• Ji-Weber 2010-2015, studied locally symmetric spaces of rank 1 and higher rank, proving that the L^p spectrum for the Laplacian on functions contains a parabolic region.

The L^p spectrum

Theorem 1 (C.-Lu)

Over the Hyperbolic space $\mathbb{H}^n$, for $0 \le k \le (n-1)/2$ *and* $1 \le p \le \infty$

$$\sigma(p,k) = Q_{p,k} = \left\{ \frac{(n-1-2k)^2}{4} + z^2 \,\middle|\, z \in \mathbb{C} \;\; \textit{with} \;\; |\mathrm{Im}\, z| \le (n-1)\left|\frac{1}{p} - \frac{1}{2}\right| \right\}$$

In the case when n is even and $k = n/2$, $\sigma(p, n/2)$ is the parabolic region $Q_{p,n/2}$ together with the point $\{0\}$.

The remaining cases are given by duality as $\sigma(p,k) = \sigma(p, n-k)$ for $n/2 \le k \le n$.

Note that $Q_{p,k} = Q_{p^*,k}$ whenever $1/p + 1/p^* = 1$.

For a fixed p the region increases in k for $1 \le k \le n/2$.

The parabolic region reduces to $[(n-1-2k)^2/4, \infty) \subset \mathbb{R}$ when $p = 2$, recovering the L^2 spectrum for k-forms computed by Donnelly.

The L^p spectrum - Some remarks

- There are always two steps to such proofs, for $\mathbb{H}^n$ but also in the general case.

- First show that a region is in the spectrum by finding approximate eigenforms.

Requires a more rigid structure at infinity so that we can write the Laplacian in some nice coordinates, and obtain a large class of approximate eigenforms. Examples are difficult to obtain.

- For the other containment, an analytical argument is required:

Find sufficient conditions on the manifold so that the resolvent operator $(\Delta - \xi)^{-1}$ is bounded for ξ in the complement of the target spectrum.

Usually, one uses resolvent estimates to show that $(\Delta - \xi)^{-1}$ is bounded on L^1 for a region in the L^1 resolvent, and then interpolates between the operators on L^1 and L^2 to get the resolvent set for any p.

- For for $\mathbb{H}^n$ the half dimension, $k = n/2$, is treated separately.

The L^p spectrum - Some remarks

• From analytical results, the resolvent operator for the Laplacian on L^p-functions is bounded for all ξ in the left-half plane without any assumptions on curvature.

• Davies-Simon-Taylor and later Taylor show that the L^p resolvent is bounded in the complement of a parabola over certain manifolds with bounded curvature. They use the wave kernel method to obtain estimates for the resolvent. This requires controlling its parametrix in a uniform way. As a result, they need to assume that the manifold has bounded geometry, and in particular injectivity radius uniformly bounded below.

• By combining the wave kernel and heat kernel method we obtain estimates for the resolvent that only assume Ricci curvature bounded below (and Weitzenböck tensor bounded below for the case of forms), allowing us to remove the assumption of bounded geometry.

Heat kernel estimates

From now on, M is a complete noncompact manifold of dimension n.

Lemma 2

Suppose that the Ricci curvature and Weitzenböck tensor on k-forms are bounded below. Then the heat kernel on k-forms satisfies

$$|\vec{h}_t(x,y)| \leq C_1 \operatorname{vol}(B_y(\sqrt{t}))^{-1} e^{C_3\, t}\ e^{-\frac{d^2(x,y)}{C_2 t}},$$

If in addition the Weitzenböck tensor on $(k+1)$-forms (resp. on $(k-1)$-forms) is bounded below, then

$$|d\vec{h}_t(x,y)| \leq C_1\, t^{-1/2} \operatorname{vol}(B_y(\sqrt{t}))^{-1} e^{C_3 t}\ e^{-\frac{d^2(x,y)}{C_2 t}}$$

$$|\delta\vec{h}_t(x,y)| \leq C_1\, t^{-1/2} \operatorname{vol}(B_y(\sqrt{t}))^{-1} e^{C_3 t}\ e^{-\frac{d^2(x,y)}{C_2 t}}$$

(resp.) where the C_i only depend on n and the curvature bounds.

Heat kernel estimates

The heat kernel estimate follows from the domination property

$$|\vec{h}_t(x,y)| \leq e^{K_1 t} h(t,x,y)$$

whenever $\mathcal{W}_k \geq -K_1$, and where h is the heat kernel on functions. It is a consequence of Kato's inequality, which allows us to compare pointwise the inner product for the Laplacian on forms, to that for functions (Hess, Schrader, Uhlenbrock).

For the derivatives of the kernel, the Bochner formula for forms gives

$$\Delta|d\vec{h}_t(x,y)| + \partial/\partial t|d\vec{h}_t(x,y)| \leq K_2|d\vec{h}_t(x,y)|,$$

whenever $\mathcal{W}_{k+1} \geq -K_2$ (resp. for $\delta\vec{h}_t$), and then we can apply the parabolic version of the Moser inequality (Cheng-Li-Yau).

Resolvent estimates

Theorem 3 (C.-Lu)

Suppose that M has Ricci curvature and Weitzenböck tensor on k-forms bounded below. Then for any $m > 0$ and $\xi \in \mathbb{R}$ large enough

$$(\Delta + \xi^2)^{-m}$$

is bounded on $L^p(\Lambda^k)$ for any $1 \leq p \leq \infty$.

If in addition the Weitzenböck tensor tensor on $(k+1)$-forms (resp. $(k-1)$-forms) is bounded below, then for any $m > 1/2$

$d(\Delta + \xi^2)^{-m}$, resp. $\delta(\Delta + \xi^2)^{-m}$, are bounded on $L^p(\Lambda^k)$.

Proof: The resolvent kernel is given by $\vec{g}_{m,\xi}(x,y) = c_n \int_0^\infty t^{m-1} e^{-\xi^2 t}\, \vec{h}_t(x,y)\, dt$

Using our heat kernel bounds we get (after some careful estimates):

$$\sup_x \int_M |\vec{g}_{m,\xi}(x,y)|dy + \sup_y \int_M |\vec{g}_{m,\xi}(x,y)|dx \leq C.$$

Schur's test now gives that the resolvent is bounded on L^p.

Resolvent estimates

In fact, using a generalized Schur's test, we can prove the following:
Let

$$\varphi(x) = (\mathrm{vol}(B_x(1)))^{-1/2}.$$

Then for any $1 \leq p, q, r^* \leq \infty$ such that $1 + \frac{1}{q} = \frac{1}{p} + \frac{1}{r^*}$, the operator

$$\varphi^{1/r^*-1}(\Delta + \xi^2)^{-m}\varphi^{1/r^*-1}$$

is bounded from L^p to L^q whenever $m > \frac{n}{2}(1 - \frac{1}{r^*})$.

The previous theorem corresponded to $r^* = 1$. In general, $q \geq p$.
The volume weight was 'necessary' in order to be able to prove the integrability of the kernels using volume comparison.

In particular, our estimates imply that certain weighted integrals of the resolvent kernel decay exponentially in $d(x, y)$.

The L^p spectrum

The *exponential rate of volume growth* of M, denoted by γ, is the infimum of all real numbers satisfying the property: for any $\varepsilon > 0$, there is a constant $C(\varepsilon)$, depending only on ε and the dimension of the manifold, such that for any $x \in M$ and any $R \geq 1$, we have

$$\mathrm{vol}(B_x(R)) \leq C(\varepsilon)\mathrm{vol}(B_x(1))e^{(\gamma+\varepsilon)R}.$$

Theorem 4 (C. - Lu)

Suppose that M has Ricci curvature and Weitzenböck tensor on k-forms bounded below. Denote by γ the exponential rate of volume growth of M and λ_1 the infimum of $\sigma(2,k)$. Let z be a complex number such that $|\mathrm{Im}(z)| > \gamma \left|\frac{1}{p} - \frac{1}{2}\right|$. Then

$$(\Delta - \lambda_1 - z^2)^{-1}$$

is a bounded operator on $L^p(\Lambda^k(M))$.

The L^p spectrum: Proof of Theorem 4

• Our proof is a modification of the wave kernel technique of Taylor. Instead of using the parametrix for the wave kernel we use a power of the resolvent.

• Let $H = \Delta - \lambda_1$, which is nonnegative on L^2.
Let $\xi > 0$ a large enough real number. For any integer $m \geq 1$, we have the resolvent identity

$$(H - z^2)^{-1} = K + (\xi^2 + z^2)^{-m}(H - z^2)^{-1}(H + \xi^2)^{-m},$$

where

$$K = (H + \xi^2)^{-1} + \cdots + (\xi^2 + z^2)^{m-1}(H + \xi^2)^{-m}.$$

Since $\xi \in \mathbb{R}$, K is bounded on L^1 for ξ large (by Theorem 3).

• It suffices to show that $(H - z^2)^{-1}(H + \xi^2)^{-m}$ is bounded on L^1.

The L^p spectrum: Proof of Theorem 4 -Key Lemma

- Let $\vec{g}_{m,\xi}$ and $S^z(x,y)$ be the kernels of $(H+\xi^2)^{-m}$ and $(H-z^2)^{-1}(H+\xi^2)^{-m}$ respectively. Then

$$S^z(x,y) = (H-z^2)^{-1}\vec{g}_{m,\xi}(x,y) = (-iz)^{-1}\int_0^\infty e^{izt}\cos(t\sqrt{H})\vec{g}_{m,\xi}(x,y)dt.$$

- Key Lemma: Under our geometric assumptions, for $z = \alpha + i(\gamma/2 + \varepsilon_o)$ with $\varepsilon_o > 0$: $\quad \sup_{y\in M}\int_M |S^z(x,y)|dx \le C < \infty.$

- Hence the operator $(H-z^2)^{-1}(H+\xi^2)^{-m}$ is bounded on L^1 whenever $\mathrm{Im}(z) > \gamma/2$.

- By the resolvent equation, $(H-z^2)^{-1}$ is bounded on L^1 for $\mathrm{Im}(z) > \gamma/2$. Replacing z with $-z$ we get Theorem 4 in the case $p=1$ for $|\mathrm{Im}(z)| > \gamma/2$.

- At the same time, $(H-z^2)^{-1}$ is a bounded operator on L^2 whenever $|\mathrm{Im}\, z| > 0$.

- Theorem 4 now follows for L^p from the Stein Interpolation Theorem (for each p we get an interpolated parabolic region).

The L^p spectrum: Proof of the Key Lemma

• The resolvent kernel $\vec{g}_{m,\xi}(x, y)$ replaces the parametrix in the classical wave kernel method.

• To prove the Key Lemma we split the resolvent kernel into two parts, one in a small neighborhood of y, and one outside.

We then use the finite propagation speed of the wave operator, our generalized resolvent estimates and the assumption on the exponential rate of volume growth to prove the estimate.

The L^p spectrum

A remark on the exponential rate of volume growth

• Let $x_o \in M$ be a fixed point. γ is bounded below by the volume entropy, that is,

$$\gamma \geq \limsup_{R\to\infty} \frac{\log \mathrm{vol}(B_{x_o}(R))}{R}$$

if $\mathrm{vol}(M) = \infty$, and

$$\gamma \geq -\limsup_{R\to\infty} \frac{\log(\mathrm{vol}(M) - \mathrm{vol}(B_{x_o}(R)))}{R}$$

if $\mathrm{vol}(M) < \infty$.

• The case $\gamma = 0$ corresponds to the concept of uniformly subexponential volume growth for a manifold introduced by Sturm.
Recall that if $\gamma = 0$ and the Ricci curvature and the Weitzenböck tensor on k-forms are bounded below, then $\sigma(p, k)$ is p-independent.

The L^p spectrum: A generalization

• Taylor: Certain functions of the Laplacian (depending on their decay rate at infinity) which are holomorphic on a strip in the complex plane, will be holomorphic on L^p.
For the proof, he considers the Fourier transform of f, and obtains estimates via the wave kernel. Bounded geometry is a requirement on M.

• We consider functions for which we can define the Laplace transform.
Let $\tilde{g}(t)$ be the inverse Laplace transformation of g. Then the kernel $G(x,y)$ of $g(\Delta)$ can be written as

$$G(x,y) = \int_0^\infty \tilde{g}(t)\vec{h}(x,y,t)dt.$$

• **Example.** Let z be a fixed complex number such that $|\mathrm{Im}(z)| > \gamma_o/2 + 2\varepsilon_o$. Let

$$g(w) = \frac{1}{w - z^2}, \qquad f(w) = \frac{1}{w^2 - z^2}.$$

Note that the inverse Laplace transform of g exists for $w > |z^2|$.
f, g satisfy the following more general assumptions:

The L^p spectrum: A generalization

1. Let γ be the exponential rate of volume growth of M. Consider the horizontal strip

$$W = \{w \in \mathbb{C} \ \mid \ |\mathrm{Im}(w)| < \gamma/2 + \varepsilon_o\}$$

for some $\varepsilon_o > 0$. Let $f(w)$ be an even holomorphic function on W satisfying

$$|f^{(j)}(w)| \leq \frac{C_j}{(1+|w|)^j}$$

for all $0 \leq j \leq n/2 + 2$ and $w \in W$.

2. Let $g(w) = f(\sqrt{w})$ and $c \gg 0$. Assume that the inverse Laplace transform $\tilde{g}(t)$ of $g(s)$ exists for $s \geq c$, that is, for any real number $s \geq c$, we have

$$g(s) = \int_0^\infty e^{-st}\tilde{g}(t)dt.$$

We further assume that $\tilde{g}$ is of at most exponential growth,

$$|\tilde{g}(t)| \leq c_1 e^{c_2 t} \quad \text{for} \quad \text{all} \quad t \in [0, \infty)$$

for constants $c_1, c_2 > 0$.

The L^p spectrum: A generalization

Theorem 5 (C. -Lu)

Let M be a manifold with exponential rate of volume growth $\gamma \le \gamma_o$ and such that its Ricci curvature and the Weitzenböck tensor on k-forms are bounded below. Suppose that f, g satisfy the above assumptions (1), (2), and let $L = \sqrt{\Delta - \lambda_1}$. Then $f(L)$ is a bounded operator on $L^1(\Lambda^k(M))$.

For any positive integer N, by Taylor's formula for $\alpha > 0$ we have

$$\begin{aligned} f(L) &= g(\Delta - \lambda_1) = A_N + B_N \\ &= \sum_{j=0}^{N-1} (-1)^j \frac{\alpha^j}{j!} g^{(j)}(\Delta - \lambda_1 + \alpha) + \frac{(-1)^N \alpha^N}{(N-1)!} \int_0^1 g^{(N)}(\Delta - \lambda_1 + t\alpha) t^{N-1} dt. \end{aligned}$$

We use the functional analytic properties of the Laplace transform and our heat kernel estimates to prove that A_N is bounded on L^1.
For B_N we use the wave kernel method, with parametrix a power of the resolvent.

The L^p spectrum: Conformally compact manifolds

Let (M, g) be a conformally compact manifold of dimension n. Borthwick shows that there exists a boundary defining function x and a compact set K such that

$$M \setminus K \cong (0, x_1) \times Y \quad \text{with} \quad g = \frac{dx^2}{\alpha(y)^2 x^2} + \frac{h(x, y, dy)}{x^2} + O(x^\infty).$$

Here, Y^{n-1} is a compact manifold (not necessarily connected), the boundary at infinity.
$h(x, y, dy)$ is a smooth family of metrics on Y and converges uniformly to a fixed smooth metric $h(0, y, dy)$ on Y as $x \to 0$.

The sectional curvature tends to $-\alpha(y)^2$ as $x \to 0$.
Suppose that $|\alpha(y)| \in [\alpha_0, \alpha_1]$.

Mazzeo proved that $\sigma_{\text{ess}}(2, k) = [\alpha_0^2(n - 2k - 1)^2/4, \infty)$ for $k < n/2$ and $\sigma_{\text{ess}}(2, n/2) = \{0\} \cup [\alpha_0^2/4, \infty)$.

However, $\gamma = (n - 1)\alpha_1$, is the exponential rate of volume growth of M.

The L^p spectrum: Conformally compact manifolds

Theorem 6 (C.-Rowlett)

Let M be a conformally compact manifold and $A = \cup_{y \in Y} |\alpha(y)|$. Then, for $1 \leq p \leq \infty$.

$$\sigma(p, 0) \supset \bigcup_{\alpha \in A} \left\{ \frac{\alpha^2 (n-1)^2}{4} + z^2 \,\middle|\, z \in \mathbb{C} \;\; \textit{with} \;\; |\operatorname{Im} z| \leq (n-1)\alpha \left| \frac{1}{p} - \frac{1}{2} \right| \right\}$$

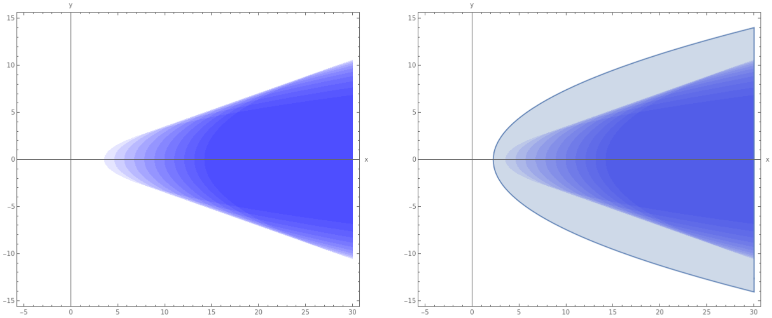
Moreover, the spectrum is contained in the parabolic region

$$\left\{ \lambda_1 + z^2 \,\middle|\, z \in \mathbb{C} \;\; \textit{with} \;\; |\operatorname{Im} z| \leq (n-1)\alpha_1 \left| \frac{1}{p} - \frac{1}{2} \right| \right\}$$

The case of the hyperbolic space corresponds to $A = \{1\}$.
In general, if $A = \{\alpha_0\}$ and $\lambda_1 = \alpha_0^2 (n-1)^2/4$, then the spectrum is precisely a parabolic region.

Note that the parabolas Q_p^α are more open as α increases and move to the right.

The L^p spectrum: Conformally compact manifolds



- The enveloping curve is a conic region.

- There are two obstructions in knowing the L^p spectrum precisely: the isolated eigenvalues of finite multiplicity which make λ_1 move to the left, and the variation of the sectional curvature at infinity.

The L^p spectrum: Other applications

• Siasos proved that if M is a warped product at infinity with sectional curvature approaching $-\alpha_0^2$ at infinity, then for $0 \leq k \leq n/2$

$$\sigma(p,k) \supset \left\{ \frac{\alpha_0^2(n-1-2k)^2}{4} + z^2 \,\middle|\, z \in \mathbb{C} \;\text{ with }\; |\operatorname{Im} z| \leq (n-1)\alpha_0 \left| \frac{1}{p} - \frac{1}{2} \right| \right\} = Q_{p,k}$$

We also have an analogous parabola which contains the spectrum, and with vertex at λ_1.

• We can control the set of isolated eigenvalues of finite multiplicity over certain quotients of hyperbolic space as in the case of the Laplacian on functions (Davies, Simon, Taylor).

Let $M = \mathbb{H}^n/\Gamma$ with Γ a geometrically finite group such that M has infinite volume and no cusps. For $0 \leq k < n/2$

$$\sigma(p,k) = \{\lambda_1, \lambda_2, \ldots, \lambda_N\} \cup Q_{p,k}$$

$\alpha_0 = 1$ and $\{\lambda_1, \lambda_2, \ldots, \lambda_N\}$ are the isolated eigenvalues of finite multiplicity for the Laplacian on $L^2(\Lambda^k)$.

The L^p spectrum -Lack of duality

• Key observation: Even if $\sigma(p,k) = \sigma(p^*,k)$ the nature of the points in the spectra can differ.

• Taylor 1989, proved that over symmetric spaces of noncompact type, every point inside a parabolic region is an eigenvalue for the L^p-spectrum of the Laplacian on functions for $p > 2$.

• Ji and Weber 2010, proved that over locally symmetric spaces of rank 1, for $p > 2$, $\sigma(p,0)$ contains an open subset of $\mathbb{C}$ in which every point is an eigenvalue, whereas for $1 < p < 2$ the set of eigenvalues is a discrete set.

-They also showed in 2015 that over certain non-compact locally symmetric spaces of higher rank and with *finite volume* every point in a parabolic region, except for a discrete set, is an eigenvalue for $1 < p < 2$.

• For the Laplacian on k-forms over $\mathbb{H}^n$ this is also the case: For $p > 2$ every point in the interior of the parabola $Q_{p,k}$ is an L^p eigenvalue, whereas for $1 \leq p \leq 2$, none of the points in $Q_{p,k}$ is an eigenvalue [C.-Lu].

Analytical properties of the L^p spectrum

• Using the duality of Banach spaces, we prove that:
$\lambda \in \sigma(p, k)$ if and only if for any $\varepsilon > 0$,
either there is an $\omega \in L^p(\Lambda^k)$ such that $\|\Delta\omega - \lambda\omega\|_{L^p} \leq \varepsilon \|\omega\|_{L^p}$
or, there is an $\omega \in L^{p^*}(\Lambda^k)$ such that $\|\Delta\omega - \lambda\omega\|_{L^{p^*}} \leq \varepsilon \|\omega\|_{L^{p^*}}$.

• Our resolvent estimates allow us to show the following:

Theorem 7 (C.-Lu)

Suppose that M has Ricci curvature and Weitzenböck tensor on k-forms bounded below, and the volume of geodesic balls of radius one is uniformly bounded below. Fix $p \geq 2$. Then, $\lambda \in \sigma(p, k)$ if and only if for any $\varepsilon > 0$ there is an $\omega \in L^p(\Lambda^k)$ such that

$$\|\Delta\omega - \lambda\omega\|_{L^p} \leq \varepsilon \|\omega\|_{L^p}.$$

Analytical properties of the L^p spectrum

Theorem 8 (C.-Lu)

Suppose that M^n is a complete manifold with Ricci curvature and the Weitzenböck tensor on $(k-1)$, k and $(k+1)$-forms bounded below. Then,

$$\sigma(p,k) \setminus \{0\} \subset \sigma(p,k-1) \cup \sigma(p,k+1)$$

for any $1 \leq p \leq \infty$.

Proof: Let $\lambda \in \rho(p,k-1) \cap \rho(p,k+1)$, $\lambda \neq 0$.

By the resolvent equation, for $\alpha > 0$ large enough,

$$(\Delta - \lambda)^{-1} = (\Delta + \alpha)^{-1} + \cdots + (\lambda + \alpha)^{2m-1}(\Delta + \alpha)^{-2m}$$
$$- \frac{(\lambda+\alpha)^{2m}}{\lambda}(\Delta + \alpha)^{-2m} + \frac{(\lambda+\alpha)^{2m}}{\lambda}(\Delta - \lambda)^{-1}\Delta(\Delta+\alpha)^{-2m}.$$

So it suffices to prove that the last operator on the right in bounded on L^p.

Analytical properties of the L^p spectrum

Observe that,

$$(\Delta-\lambda)^{-1}\Delta(\Delta+\alpha)^{-2m} = d(\Delta+\alpha)^{-m}(\Delta-\lambda)^{-1}\delta(\Delta+\alpha)^{-m} + \delta(\Delta+\alpha)^{-m}(\Delta-\lambda)^{-1}d(\Delta+\alpha)^{-m}.$$

The operators $d(\Delta+\alpha)^{-m}$ and $\delta(\Delta+\alpha)^{-m}$ are bounded on L^p under our curvature assumptions, for any $m > 1/2$ and α large enough.

So, if $\lambda \in \rho(p, k-1) \cap \rho(p, k+1)$, $\lambda \neq 0$, the operator on the right side is bounded on $L^p(\Lambda^k(M))$, and in consequence $\lambda \in \rho(p, k)$ by the resolvent equation above.

• This result was necessary for the computation of $\sigma(p, n/2)$ over $\mathbb{H}^n$:
Note that $\{0\} \in \sigma(p, n/2)$, for p near $p = 2$, and there is a spectral gap so the interpolation argument does not work well to give the resolvent parabola.
Observing that $Q_{p,(n-2)/2} = Q_{p,(n+2)/2} = Q_{p,n/2}$, Theorem 8 allows us to show that $\sigma(p, n/2) \setminus \{0\} = Q_{p,n/2}$.

Thank you!